Lecture : DTMCs : Classification, Reversibility

Last time, we ended with "Big Theorem for Markov Chains":

connects classification of states w long-term behavior of MCs.

<u>Thm:</u> Let $(X_n)_{n\geq 0}$ be irreducible MC. Exactly 1 of the following is true:

can go back & forth btwn 2 states

① All states are transient/null-recurrent. No SD exists and

each state visited finite # of times

re-visits each state only often but ∞ time to return to any given state

stationary distribution

$$\lim_{n\to\infty} P_{ij}^{\,n} = 0 \qquad \forall i,j \in S$$

② All states positive recurrent. SD π exists & is unique & satisfies:

$$\pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{\,k} = \frac{1}{\underbrace{\mathbb{E}[T_j \mid X_0 = j]}_{\text{expected amt of time to return to state j given you start in state j.}}} \qquad \forall i,j \in S$$

Moreover, if chain is aperiodic, then

$$P_{ij}^{\,n} \longrightarrow \pi_j \quad \text{as } n\to\infty \quad \forall i,j \in S$$

<u>Example:</u> Every irreducible finite state Markov chain is positive recurrent and therefore has a unique SD.

<u>Pf</u>: Suppose ① holds:

$$1 = \sum_{j\in S} P_{ij}^{\,n} \longrightarrow 0 \quad \text{as } n\to\infty$$

(can exchange limit & finite sum)

$\rightarrow 1 = 0$ contradiction ∎

Ex: Let $G=(V,E)$ be a finite connected graph & consider a random walk on G defined by:

$$P_{ij} = \begin{cases} \frac{1}{d(i)} & (i,j)\in E \\ 0 & o/w \end{cases}$$

$d(i)$ = degree of vertex i.

Q/ Given I start in state i, what's the expected # of steps until I return to state i?

A/ Connected graph $\Rightarrow$ irreducible MC

Finite graph $\Rightarrow$ positive recurrent

We should solve for the SD π!

$$\pi_j = \sum_i P_{ij} \pi_i = \underbrace{\sum_{i=(i,j)\in E} \frac{1}{d(i)} \pi_i}_{d(j) \text{ terms in sum}}$$

Guess: $\pi_i = \alpha d(i)$, $i \in S$

Check:

$$\alpha d(j) = \sum_{i=(i,j)\in E} \frac{1}{d(i)} \alpha d(i) = \alpha d(j) \quad ✓$$

Q/ what is α?

$$1 = \sum_j \pi_j = \alpha \sum_j d(j) = \alpha 2|E|$$

$$\Rightarrow \pi_j = \frac{d(j)}{2|E|}$$

$$\Rightarrow \mathbb{E}[T_j \mid X_0 = j] = \frac{2|E|}{d(j)}$$

Ex: Let $(X_n)_{n\geq 0}$ be irreducible pos. recurrent MC.

Let $r: S \to \mathbb{R}$ be a "reward fcn" and assume I collected reward $r(j)$ every time I enter state j, $j \in S$.

The process: $r(X_n)_{n\geq 0}$ is a Markov "reward process"

Q/ what is long-term average reward I receive?

A/ $\frac{1}{n}\sum_{k=1}^{n} r(X_k) \approx$?? for large n?

It turns out:

$$\frac{1}{n}\sum_{k=1}^{n} r(X_k) \longrightarrow \mathbb{E}\, r(X) \quad \text{a.s.}$$

where $X \sim \pi$.

$\leadsto$ Ergotic Thm for MCs.

Pf:

$$\frac{1}{n}\sum_{k=1}^{n} r(X_k) = \sum_{j\in S} \frac{N_j(n)}{n} r(j) \xrightarrow{a.s.} \sum_j \frac{1}{\mathbb{E}[T_j \mid X_0 = j]} r(j)$$

$$\overset{\text{Big Thm}}{=} \sum_j \pi_j r(j)$$

$N_j(n)$ = # times I've entered state j up to time n.

$$\frac{n}{N_j(n)} \sim \frac{T_j^{(1)} + \cdots + T_j^{(N_j(n))}}{N_j(n)} \xrightarrow[\text{SLLN}]{} \mathbb{E}[T_j \mid X_0 = j] \quad a.s.$$

where $T_j^{(i)} \overset{\sim}{\text{iid}} T_j$

General strategy for applying Big Thm to irreducible MC is to try to compute SD to see which alternative you're in.

Q / How do we compute SD in general?

A / Just appeal to the defining equations:

$$\underbrace{\pi_j}_{\text{flow out of state } j} = \underbrace{\sum_i \pi_i P_{ij}}_{\text{flow into state } j} \quad \forall j \text{ (Balance Eqns)}$$

Recall: This is like pumps in 16A!

In many practical cases, computation can be simplified if chain is reversible.

Def: An irreducible MC is reversible if $\exists$ a probability vector π satisfying

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i,j \in S \quad \text{(Detailed Balance Eqns)}$$

Rmks: If MC is reversible, then π is SD.

DBE $\Rightarrow$ BE (sum both sides of DBE over j)

$$\pi_i = \sum_j \pi_i P_{ij} = \sum_j \pi_j P_{ji} \Rightarrow \pi = \pi P$$

- The term "reversible" come from fact that if

$X_0 \sim \pi$ then

$$(X_0 \cdots X_n) \overset{d}{=} (X_n \cdots X_0)$$

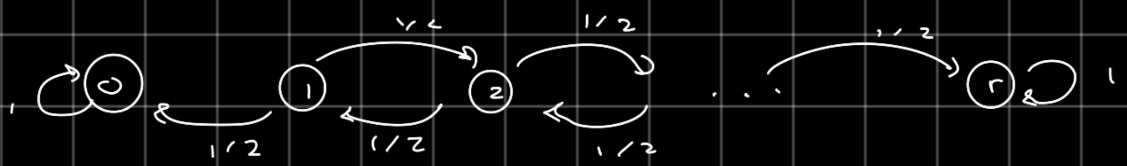
· Many chains in practice are reversible

So, a general strategy for computing SD:

1. Hope your chain is reversible & try to solve DBEs
2. If that doesn't work, try to solve BE
3. " " " " just give up bc SD doesn't exist

Other questions are interesting but not addressed by Big Thm:

Ex: Gambler's ruin (not irreducible)



Q_1/ What's the prob that we win \$R given initial capital \$k? ("hitting probability")

Q_2/ What's the expected length of the game? ("hitting time")

Both questions can be solved w/general technique called "First Step Analysis"

(ie analyze MC on step-by-step basis)

Hitting Times: Let $A \subset S$ and define hitting time

$$T_A = \min\{n \geq 0 : X_n \in A\}$$

Distribution of T_A likely out of reach but we can compute $\mathbb{E}[T_A \mid X_0 = i]$

<u>Strategy</u>: Define $t(i) := \mathbb{E}[T_A \mid X_0 = i]$

Note: $t(i) = 0$ if $i \in A$

If $i \notin A$, then:

$$t(i) = \mathbb{E}[T_A \mid X_0 = i] \overset{\text{first-step}}{=} 1 + \sum_j P_{ij}\, \mathbb{E}[T_A \mid X_0 = j]$$

$$= 1 + \sum_j P_{ij}\, t(j)$$

system of linear equations for computing hitting times
$\Rightarrow$

$$\boxed{\begin{array}{ll} t(i) = 0 & i \in A \\ t(i) = 1 + \sum_j P_{ij}\, t(j) & i \notin A \end{array}}$$

unknowns are $t(i)$'s

will have soln if $T_A < \infty$ a.s.

Ex: Gambler's ruin. Time until game ends.

we take $A = \{0, R\}$

$$t(i) = \begin{cases} 0 & i \in \{0, R\} \\ 1 + \frac{1}{2}(t(i-1) + t(i+1)) & i \in \{1, \cdots, R-1\} \end{cases}$$

soln: $t(k) = k(R-k)$

expected duration of game given initial capital \$K

Hitting Probabilities are similarly computed.

For $a, b \in S$, define

$$T_a = \min\{n \geq 0 : X_n = a\}$$

$$T_b = \min\{n \geq 0 : X_n = b\}$$

and assume $\min\{T_a, T_b\} < \infty$ a.s. given any starting state.

Q/ what's the prob I hit a before b?

strategy: Define $h(i) = P\{T_a < T_b \mid X_0 = i\}$

Observe: $h(a) = 1$, $h(b) = 0$

For $i \notin \{a, b\}$

$$\begin{aligned} h(i) &= P\{T_a < T_b \mid X_0 = i\} \\ &\overset{\text{1st step}}{=} \sum_j P_{ij}\, P\{T_a < T_b \mid X_1 = j, X_0 = i\} \\ &\overset{\text{by Markovity}}{=} \sum_j P_{ij}\, P\{T_a < T_b \mid X_0 = j\} \\ &= \sum_j P_{ij}\, h(j) \end{aligned}$$

Systems of eqns for hitting probs:

$h(a) = 1 \qquad h(b) = 0$

$$h(i) = \sum_{j \in S} h(j) P_{ij} \qquad i \notin \{a, b\}$$

Ex: Gambler's ruin

$a = R$

$b = 0$

$h(R) = 1 \qquad h(0) = 0$

$$h(i) = \frac{1}{2}\left[h(i-1) + h(i+1)\right] \qquad i \in \{1, \dots, R-1\}$$

soln: $h(k) = \dfrac{k}{R} \qquad 0 \le k \le R$